

An analysis of the equational properties of the well-founded fixed point

Arnaud Carayol
Université Paris Est and CNRS
France

Zoltán Ésik*
University of Szeged
Hungary

December 2, 2015

Abstract

Well-founded fixed points have been used in several areas of knowledge representation and reasoning and to give semantics to logic programs involving negation. They are an important ingredient of approximation fixed point theory. We study the logical properties of the (parametric) well-founded fixed point operation. We show that the operation satisfies several, but not all of the equational properties of fixed point operations described by the axioms of iteration theories.

1 Introduction

Fixed points and fixed point operations have been used in just about all areas of computer science. There has been a tremendous amount of work on the existence, construction and logic of fixed point operations. It has been shown that most fixed point operations, including the least (or greatest) fixed point operation on monotonic functions over complete lattices, satisfy the same equational properties. These equational properties are captured by the notion of iteration theories, or iteration categories, cf. [3] or the recent survey [17].

For an account of fixed point approaches to logic programming containing original references we refer to [21]. These approaches, and in particular the stable and well-founded fixed point semantics of logic programs with negation, based on the notion of bilattices, have led to the development of an elegant abstract ‘approximation fixed point theory’, cf. [8, 9, 27].

In this paper, we study the equational properties of the well-founded fixed point operation as defined in [8, 9, 27] with the aim of relating well-founded fixed points to iteration categories. We extend the well-founded fixed point operation to a parametric operation giving rise to an external fixed point (or dagger) operation [3, 4] over the cartesian category of approximation function pairs between complete bilattices. We offer an initial analysis of the equational properties of the well-founded fixed point operation. Our main results show that several identities of iteration theories hold for the well-founded fixed point operation, but some others fail.

*The second author received support from NKFI grant no. ANN 110883 and the Université Paris Est.

2 Complete lattices and bilattices

Recall that a *complete lattice* [6] is a partially ordered set $L = (L, \leq)$ such that each $X \subseteq L$ has a supremum $\bigvee X$ and hence also an infimum $\bigwedge X$. In particular, each complete lattice has a least and a greatest element, respectively denoted either \perp and \top , or 0 and 1. We say that a function $f : L \rightarrow L$ over a complete lattice L is monotonic (anti-monotonic, resp.) if for all $x, y \in L$, if $x \leq y$ then $f(x) \leq f(y)$ ($f(x) \geq f(y)$, resp.).

A *complete bilattice*¹ [21, 22, 23] (B, \leq_p, \leq_t) is equipped with two partial orders, \leq_p and \leq_t , both giving rise to a complete lattice. We will denote the \leq_p -least and greatest elements of a complete bilattice by \perp and \top , and the \leq_t -least and greatest elements by 0 and 1, respectively.

An example, depicted in Figure 1, of a complete bilattice is **4**, which has 4 elements, $\perp, \top, 0, 1$. The nontrivial order relations are given by $\perp \leq_p 0, 1 \leq_p \top$ and $0 \leq_t \perp, \top \leq_t 1$.

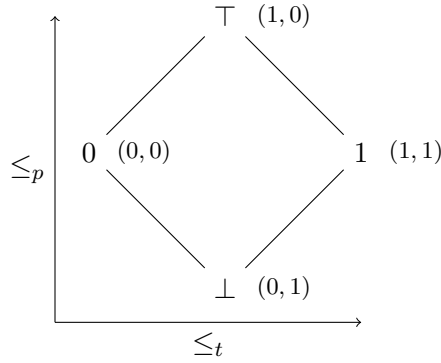


Figure 1: A representation of $\mathbf{4} \approx \mathbf{2} \times \mathbf{2}$ taken from [21].

Two closely related constructions of a complete bilattice from a complete lattice are described in [8] and [22], see [23] for the origins of the constructions. Here we recall one of them. Suppose that $L = (L, \leq)$ is a complete lattice with extremal (i.e., least and greatest) elements 0 and 1. Then define the partial orders \leq_p and \leq_t on $L \times L$ as follows:

$$\begin{aligned} (x, x') \leq_p (y, y') &\Leftrightarrow x \leq y \wedge x' \geq y' \\ (x, x') \leq_t (y, y') &\Leftrightarrow x \leq y \wedge x' \leq y'. \end{aligned}$$

Then $L \times L$ is a complete bilattice with \leq_p -extremal elements $\perp = (0, 1)$ and $\top = (1, 0)$, and \leq_t -extremal elements $0 = (0, 0)$ and $1 = (1, 1)$. Note that when L is the 2-element lattice $\mathbf{2} = \{0 \leq 1\}$, then $L \times L$ is isomorphic to **4**. In this paper, we will mainly be concerned with the ordering \leq_p .

In any category, we usually denote the composition of morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ by $g \circ f$ and the identity morphisms by id_A . We let SET denote the category of sets and functions and we denote by CL the category of complete lattices and monotonic functions. Both SET and CL have all products and hence are *cartesian categories*. The usual direct product, equipped with the pointwise order in CL, serves as categorical product. In CL, a terminal object is a

¹Sometimes bilattices are equipped with a negation operation and the bilattices as defined here are called pre-bilattices.

1-element lattice T . In both categories, for any sequence A_1, \dots, A_n of objects, the categorical projection morphisms $\pi_i^{A_1 \times \dots \times A_n} : A_1 \times \dots \times A_n \rightarrow A_i$, $i \in [n] = \{1, \dots, n\}$, are the usual projection functions.

Products give rise to a *tupling* operation. Suppose that $f_i : C \rightarrow A_i$, $i \in [n]$ in SET or CL, or in any cartesian category. Then there is a unique $f : C \rightarrow A_1 \times \dots \times A_n$ with $\pi_i^{A_1 \times \dots \times A_n} \circ f = f_i$ for all $i \in [n]$. We denote this unique morphism f by $\langle f_1, \dots, f_n \rangle$ and call it the (target) tupling of the f_i (or pairing, when $n = 2$).

And when $f : C \rightarrow A$ and $g : D \rightarrow B$, then we define $f \times g$ as the unique morphism $h : C \times D \rightarrow A \times B$ with $\pi_1^{A \times B} \circ h = f \circ \pi_1^{C \times D}$ and $\pi_2^{A \times B} \circ h = g \circ \pi_2^{C \times D}$.

When $m, n \geq 0$, ρ is a function $[m] \rightarrow [n]$ and A_1, \dots, A_n is a sequence of objects in a cartesian category, we associate with ρ (and A_1, \dots, A_n) the morphism

$$\rho^{A_1, \dots, A_n} = \langle \pi_{\rho(1)}^{A_1 \times \dots \times A_n}, \dots, \pi_{\rho(m)}^{A_1 \times \dots \times A_n} \rangle$$

from $A_1 \times \dots \times A_n$ to $A_{\rho(1)} \times \dots \times A_{\rho(m)}$ (Note that in SET and CL, ρ^{A_1, \dots, A_n} maps $(x_1, \dots, x_n) \in A_1 \times \dots \times A_n$ to $(x_{\rho(1)}, \dots, x_{\rho(m)}) \in A_{\rho(1)} \times \dots \times A_{\rho(m)}$.) With a slight abuse of notation, we usually let ρ denote this morphism as well. Morphisms of this form are sometimes called *base morphisms*. When $m = n$ and ρ is a bijection, then the associated morphism $A_1 \times \dots \times A_n \rightarrow A_{\rho(1)} \times \dots \times A_{\rho(n)}$ is an isomorphism. Its inverse is the morphism associated with the inverse ρ^{-1} of the function ρ . For each object A , the base morphism associated with the unique function $[m] \rightarrow [1]$ is the *diagonal morphism* $\Delta_m^A = \langle \text{id}_A, \dots, \text{id}_A \rangle : A \rightarrow A^m$, usually denoted just Δ_m .

3 Iteration categories

The category CL is equipped with an (external) *fixed point* or *dagger* operation [3, 4] mapping a monotonic function $f : A \times B \rightarrow A$ to the monotonic function $f^\dagger : B \rightarrow A$ such that for all $y \in B$, $f^\dagger(y)$ is the least solution of the fixed point equation $x = f(x, y)$. We will sometimes denote $f^\dagger(y)$ by $\mu x.f(x, y)$. It provides the unique least solution to the parametric fixed point equation

$$x = f(x, y). \tag{1}$$

When B is the terminal object T , f can be viewed as a function $A \rightarrow A$ and f^\dagger can be identified with an element of A .

The least fixed point operation † over CL satisfies several nontrivial identities captured by the notion of *iteration theories* or *iteration categories* [3, 17]. For later use, we collect here some of these identities.

FIXED POINT IDENTITY

$$f^\dagger = f \circ \langle f^\dagger, \text{id}_B \rangle,$$

where $f : A \times B \rightarrow A$.

The fixed point identity expresses that $f^\dagger(y)$ is a solution of the fixed point equation (1).

PARAMETER IDENTITY

$$(f \circ (\text{id}_A \times g))^\dagger = f^\dagger \circ g,$$

for all $f : A \times B \rightarrow A$ and $g : C \rightarrow B$.

In functional notation, the parameter identity expresses that if $h(x, z) = f(x, g(z))$, then for the least solution $h^\dagger(z)$ of the equation $x = h(x, z)$ it holds that $h^\dagger(z) = f^\dagger(g(z))$, where $f^\dagger(y)$ is the least solution of $x = f(x, y)$.

PERMUTATION IDENTITY

$$(\rho \circ f \circ (\rho^{-1} \times \text{id}_B))^\dagger = \rho \circ f^\dagger,$$

for all $f : A_1 \times \cdots \times A_n \times B \rightarrow A_1 \times \cdots \times A_n$ and permutation $\rho : [n] \rightarrow [n]$.

This can be explained alternatively as follows. Consider the (systems of) fixed point equations

$$x = f(x, z) \tag{2}$$

and

$$y = \rho(f(\rho^{-1}(y), z)), \tag{3}$$

where x ranges over $A_1 \times \cdots \times A_n$, y ranges over $A_{\rho(1)} \times \cdots \times A_{\rho(n)}$ and $z \in B$. Here, ρ also denotes the bijective function $A_1 \times \cdots \times A_n \rightarrow A_{\rho(1)} \times \cdots \times A_{\rho(n)}$ as explained above, and ρ^{-1} also denotes the inverse of this function. Then the permutation identity expresses that the least solution of (3) is $\rho(f^\dagger(z))$, where $f^\dagger(z)$ is the least solution of (2).

COMPOSITION IDENTITY

$$(f \circ \langle g, \pi_2^{A \times C} \rangle)^\dagger = f \circ \langle (g \circ \langle f, \pi_2^{B \times C} \rangle)^\dagger, \text{id}_C \rangle,$$

where $f : B \times C \rightarrow A$ and $g : A \times C \rightarrow B$.

The composition identity relates the fixed point equations

$$x = f(g(x, z), z) \tag{4}$$

and

$$y = g(f(y, z), z). \tag{5}$$

It asserts that the least solution of (4) can be obtained by applying f to the least solution of (5) and the parameter.

DOUBLE DAGGER IDENTITY

$$f^{\dagger\dagger} = (f \circ (\langle \text{id}_A, \text{id}_A \rangle \times \text{id}_B))^\dagger,$$

for all $f : A \times A \times B \rightarrow A$.

This identity means that the least solution of the equation

$$x = f(x, x, z)$$

is the same as the least solution of

$$y = f^\dagger(y, z),$$

where $f^\dagger(y, z)$ is the least solution of $x = f(x, y, z)$.

PAIRING IDENTITY

$$\langle f, g \rangle^\dagger = \langle f^\dagger \circ \langle h^\dagger, \text{id}_C \rangle, h^\dagger \rangle,$$

for all $f : A \times B \times C \rightarrow A$ and $g : A \times B \times C \rightarrow B$, where $h = g \circ \langle f^\dagger, \text{id}_{B \times C} \rangle : B \times C \rightarrow B$.

This identity was independently found in [1] and [7]. As is well-known, it asserts that a system

$$\begin{aligned} x &= f(x, y, z) \\ y &= g(x, y, z) \end{aligned}$$

can be solved by Gaussian elimination by solving the first equation and substituting the solution into the second equation to obtain

$$\begin{aligned} x &= f^\dagger(y, z) \\ y &= g(f^\dagger(y, z), y, z) = h(y, z), \end{aligned}$$

and then by solving the second equation and substituting the solution into the first to obtain the final result

$$\begin{aligned} x &= f^\dagger(h^\dagger(z), z) \\ y &= h^\dagger(z). \end{aligned}$$

In conjunction with the fixed point and parameter identities, the following is a special case of the pairing identity:

$$\langle f, g \circ (\pi_2^{A \times B} \times \text{id}_C) \rangle^\dagger = \langle f^\dagger \circ \langle g^\dagger, \text{id}_C \rangle, g^\dagger \rangle, \quad (6)$$

where $f : A \times B \times C \rightarrow A$ and $g : B \times C \rightarrow B$. In the category CL, it asserts that the least solution of the system of equations

$$\begin{aligned} x &= f(x, y, z) \\ y &= g(y, z) \end{aligned}$$

is $x = f^\dagger(g^\dagger(z), z)$ and $y = g^\dagger(z)$.

GROUP IDENTITIES

Suppose that G is a finite group whose underlying set is $[n]$. Let $i \cdot j$ denote the multiplication of $i, j \in [n]$. The identity associated with G is:

$$\langle f \circ (\rho_1 \times \text{id}_B), \dots, f \circ (\rho_n \times \text{id}_B) \rangle^\dagger = \Delta_n \circ (f \circ (\Delta_n \times \text{id}_B))^\dagger$$

where $f : A^n \times B \rightarrow A$ and for each i , ρ_i denotes the function $[n] \rightarrow [n]$ given by $j \mapsto i \cdot j$ (as well as the associated morphism $\rho_i^{A, \dots, A} = \langle \pi_{i \cdot 1}^{A^n}, \dots, \pi_{i \cdot n}^{A^n} \rangle : A^n \rightarrow A^n$ and $\Delta_n = \Delta_n^A$ is the diagonal morphism $A \rightarrow A^n$ defined above.

This identity can be explained in the following way. Consider the system of equations

$$\begin{aligned} x_1 &= f(x_{1 \cdot 1}, \dots, x_{1 \cdot n}, y) \\ &\vdots \\ x_n &= f(x_{n \cdot 1}, \dots, x_{n \cdot n}, y) \end{aligned} \tag{7}$$

and the single equation

$$x = f(x, \dots, x, y). \tag{8}$$

Then the group identity associated with G asserts that (7) is equivalent to (8) in the sense that each component of the least solution of (7) agrees with the least solution of (8).

Each finite group G (equipped with the natural self action) can be seen as a finite automaton, and in a similar fashion, one may associate an identity with every finite automaton [11]. These are essentially the commutative identities of [10].

Definition 3.1 *An iteration category is a cartesian category equipped with a dagger operation satisfying either the parameter, fixed point, pairing, permutation and group (or commutative) identities, or the parameter, composition, double dagger and group (or commutative) identities.*

The following completeness result is from [10, 3].

Theorem 3.2 *An identity involving the cartesian category operations and dagger holds in CL with the least fixed point operation as dagger iff it holds in all iteration categories.*

Remark 3.3 Iteration categories, or iteration theories, were introduced independently in [2] and [10]². The axiomatization in [10] used the commutative identities. It was proved in [11] that the commutative identities can be simplified to the group identities. Moreover, it was shown that the identities associated with the members of a subclass \mathcal{G} of the finite groups suffices instead of all group identities iff every finite group is isomorphic to a quotient of a subgroup of a group in \mathcal{G} , see [11, 13]. Nevertheless some further simplifications of the axioms are still possible, see [12, 15].

We mention one more property that is not an identity, but a quasi-identity. It is stronger than the group identities, yet most of the standard models satisfy it. (Actually the commutative identities were introduced in [10] in order to replace this quasi-identity by weaker identities, since when it comes to equational theories, the best way to present them is by providing equational bases.)

WEAK FUNCTORIAL IMPLICATION

This axiom asserts that for all $f : A^n \times B \rightarrow A^n$ and $g : A \times B \rightarrow A$, if $f \circ (\Delta_n \times \text{id}_B) = \Delta_n \circ g$, then

$$f^\dagger = \Delta_n \circ g^\dagger.$$

²In [10], iteration theories were called ‘generalized iterative theories’.

In CL, this means that if $f = \langle f_1, \dots, f_n \rangle : A^n \times B \rightarrow A^n$ and $g : A \times B \rightarrow A$ are such that $f_i(x, \dots, x, y) = g(x, y)$ for all $i \in [n]$, then the system of equations

$$\begin{aligned} x_1 &= f_1(x_1, \dots, x_n, y) \\ &\vdots \\ x_n &= f_n(x_1, \dots, x_n, y) \end{aligned}$$

is equivalent to the single equation

$$x = g(x, y).$$

It is clear that if the weak functorial implication holds, then so do the group (or commutative) identities.

Remark 3.4 Sometimes we will apply the least fixed point operation to functions $f : A \times B \rightarrow A$, where A, B are complete lattices, which are monotonic in the first argument but anti-monotonic in the second. Such a function may be viewed as a monotonic function $A \times B^d \rightarrow A$, where B^d is the dual of B . Hence, in this case, f^\dagger is a monotonic function $B^d \rightarrow A$, or –as we will consider it– an anti-monotonic function $B \rightarrow A$. More generally, we will also consider functions that are monotonic in some arguments and anti-monotonic in others, but always take the least fixed point w.r.t. an argument in which the function is monotonic.

4 The category CL

The objects of **CL** are complete lattices. Suppose that A, B are complete lattices. A morphism from A to B in **CL**, denoted $f : A \xrightarrow{\bullet} B$, is a \leq_p -monotonic function $f : A \times A \rightarrow B \times B$, where $A \times A$ and $B \times B$ are the complete bilattices determined by A and B . Thus, $f = \langle f_1, f_2 \rangle$ such that $f_1 : A \times A \rightarrow B$ is monotonic in its first argument and anti-monotonic in the second argument, and $f_2 : A \times A \rightarrow B$ is anti-monotonic in its first argument and monotonic in its second argument. (Such functions f are called approximations in [27].) Composition is ordinary function composition and for each complete lattice A , the identity morphism $\text{id}_A : A \xrightarrow{\bullet} A$ is the identity function $\text{id}_{A \times A} = \text{id}_A \times \text{id}_A = \langle \pi_1^{A \times A}, \pi_2^{A \times A} \rangle : A \times A \rightarrow A \times A$.

The category **CL** has finite products. (Actually it has all products). Indeed, a terminal object of **CL** is any 1-element lattice. Suppose that A_1, \dots, A_n are complete lattices. Then consider the direct product $A_1 \times \dots \times A_n$ as an object of **CL** together with the following morphisms $\pi_i^{A_1 \times \dots \times A_n} : A_1 \times \dots \times A_n \xrightarrow{\bullet} A_i$, $i \in [n]$. For each i , $\pi_i^{A_1 \times \dots \times A_n}$ is the function

$$A_1 \times \dots \times A_n \times A_1 \times \dots \times A_n \rightarrow A_i \times A_i$$

defined by

$$\pi_i^{A_1 \times \dots \times A_n}(x_1, \dots, x_n, x'_1, \dots, x'_n) = (x_i, x'_i),$$

so that in SET, $\pi_i^{A_1 \times \dots \times A_n}$ can be written as

$$\langle \pi_i^{A_1 \times \dots \times A_n \times A_1 \times \dots \times A_n}, \pi_{n+i}^{A_1 \times \dots \times A_n \times A_1 \times \dots \times A_n} \rangle = \pi_i^{A_1 \times \dots \times A_n} \times \pi_i^{A_1 \times \dots \times A_n}.$$

It is easy to see that the morphisms $\pi_i^{A_1 \times \dots \times A_n}$, $i \in [n]$, determine a product diagram in **CL**. To this end, let $f^i = \langle f_1^i, f_2^i \rangle : C \xrightarrow{\bullet} A_i$ in **CL**, for all $i \in [n]$, so that each f^i is a \leq_p -monotonic function $C \times C \rightarrow A_i \times A_i$. Then let $h = \langle h_1, h_2 \rangle$, where $h_1 = \langle f_1^1, \dots, f_1^n \rangle$ and $h_2 = \langle f_2^1, \dots, f_2^n \rangle$ in the category **CL**. Thus, h_1 and h_2 are functions $C \times C \rightarrow A_1 \times \dots \times A_n$.

We prove that h is the target tupling of f^1, \dots, f^n in **CL**. First, since each f_1^i is monotonic in its first argument and anti-monotonic in the second argument, the same holds for h_1 . In the same way, h_2 is anti-monotonic in the first argument and monotonic in the second. Thus, h is \leq_p -monotonic. Next, writing just π_i for $\pi_i^{A_1 \times \dots \times A_n}$ and π_i for $\pi_i^{A_1 \times \dots \times A_n}$, where $i \in [n]$, we have

$$\begin{aligned} \pi_i \circ h &= \pi_i \circ \langle h_1, h_2 \rangle \\ &= (\pi_i \times \pi_i) \circ \langle \langle f_1^1, \dots, f_1^n \rangle, \langle f_2^1, \dots, f_2^n \rangle \rangle \\ &= \langle \pi_i \circ \langle f_1^1, \dots, f_1^n \rangle, \pi_i \circ \langle f_2^1, \dots, f_2^n \rangle \rangle \\ &= \langle f_1^i, f_2^i \rangle \\ &= f_i. \end{aligned}$$

It is also clear that h is the unique morphism $C \xrightarrow{\bullet} A_1 \times \dots \times A_n$ in **CL** with this property.

Proposition 4.1 ***CL** is a cartesian category in which the product of any objects A_1, \dots, A_n agrees with their product in **CL**.*

By the above argument, the tupling of any sequence of morphisms $f^i = \langle f_1^i, f_2^i \rangle : C \xrightarrow{\bullet} A_i$ in **CL** is $h = \langle h_1, h_2 \rangle$, where h_1 is the tupling of the f_1^i and h_2 is the tupling of the f_2^i in **SET**. We will denote it by $\langle f^1, \dots, f^n \rangle : C \xrightarrow{\bullet} A_1 \times \dots \times A_n$.

For further use, we note the following. Suppose that $\rho : [m] \rightarrow [n]$ and A_1, \dots, A_n are complete lattices. Then the associated morphism $\rho^{A_1, \dots, A_n} : A_1 \times \dots \times A_n \xrightarrow{\bullet} A_{\rho(1)} \times \dots \times A_{\rho(m)}$ in **CL** is the function

$$A_1 \times \dots \times A_n \times A_1 \times \dots \times A_n \rightarrow A_{\rho(1)} \times \dots \times A_{\rho(m)} \times A_{\rho(1)} \times \dots \times A_{\rho(m)}$$

given by

$$(x_1, \dots, x_n, x'_1, \dots, x'_n) \mapsto (x_{\rho(1)}, \dots, x_{\rho(m)}, x'_{\rho(1)}, \dots, x'_{\rho(m)}).$$

Thus,

$$\rho^{A_1, \dots, A_n} = \rho^{A_1, \dots, A_n} \times \rho^{A_1, \dots, A_n},$$

where ρ^{A_1, \dots, A_n} is the morphism associated with ρ and A_1, \dots, A_n in **SET** (or **CL**). This is in accordance with $\mathbf{id}_A = \mathbf{id}_A \times \mathbf{id}_A$.

Suppose that $f : C \xrightarrow{\bullet} A$ and $g : D \xrightarrow{\bullet} B$ in **CL**, so that f is a function $C \times C \rightarrow A \times A$ and g is a function $D \times D \rightarrow B \times B$. Then $f \times g : C \times D \xrightarrow{\bullet} A \times B$ in the category **CL** is the function

$$(\mathbf{id}_A \times \langle \pi_2^{B \times A}, \pi_1^{B \times A} \rangle \times \mathbf{id}_B) \circ h \circ (\mathbf{id}_C \times \langle \pi_2^{D \times C}, \pi_1^{D \times C} \rangle \times \mathbf{id}_D) : C \times D \times C \times D \rightarrow A \times B \times A \times B,$$

where h is $f \times g : C \times C \times D \times D \rightarrow A \times A \times B \times B$ in **SET**. Hence, $h = \langle h_1, h_2 \rangle$ with

$$\begin{aligned} h_1(x, y, x', y') &= (f_1(x, x'), g_1(y, y')) \\ h_2(x, y, x', y') &= (f_2(x, x'), g_2(y, y')). \end{aligned}$$

4.1 Some subcategories

Motivated by [8, 9, 27], we define several subcategories of **CL**. Suppose that A, B are complete lattices. Following [8], we call an ordered pair $(x, x') \in A \times A$ *consistent* if $x \leq x'$. Moreover, we call $f : A \xrightarrow{\bullet} B$ in **CL** consistent if it maps consistent pairs to consistent pairs. It is clear that if $f : A \xrightarrow{\bullet} B$ and $g : B \xrightarrow{\bullet} C$ in **CL** are consistent, then so is $g \circ f : A \xrightarrow{\bullet} C$, moreover, id_A is always consistent. Also, for any sequence A_1, \dots, A_n of complete lattices, the projections $\pi_i^{A_1 \times \dots \times A_n} : A_1 \times \dots \times A_n \xrightarrow{\bullet} A_i$, $i \in [n]$ are consistent. And when $f_i : C \xrightarrow{\bullet} A_i$, for all $i \in [n]$, then $\langle f_1, \dots, f_n \rangle : C \xrightarrow{\bullet} A_1 \times \dots \times A_n$ is consistent iff each f_i is. Hence, the consistent morphisms in **CL** determine a cartesian subcategory of **CL** with the same product diagrams. Let **CCL** denote this subcategory.

We define two subcategories of **CCL**. The first one, **ACL**, is the subcategory determined by those morphisms $f = \langle f_1, f_2 \rangle : A \xrightarrow{\bullet} B$ in **CL** such that $f_1(x, x) \leq f_2(x, x)$ for all $x \in A$. The second, **EACL**, is the subcategory determined by those $f : A \xrightarrow{\bullet} B$ with $f_1(x, x) = f_2(x, x)$. These are again cartesian subcategories with the same product diagrams.

As noted in [8], most applications of approximation fixed point theory use *symmetric* functions. We introduce the subcategory of **CL** having complete lattices as object but only symmetric \leq_p -preserving functions as morphisms.

Suppose that $f : A \xrightarrow{\bullet} B$ in **CL**, say $f = \langle f_1, f_2 \rangle$. We call f symmetric if $f_2(x, x') = f_1(x', x)$, i.e., when

$$f_2 = f_1 \circ \langle \pi_2^{A \times A}, \pi_1^{A \times A} \rangle : A \times A \rightarrow B.$$

We will express this condition in a concise way as $f_2 = f_1^{\text{op}}$.

It is easy to prove that if $f : A \xrightarrow{\bullet} B$ and $g : B \xrightarrow{\bullet} C$ are symmetric, then so is $g \circ f$. Moreover, id_A is always symmetric. Thus, symmetric morphisms determine a subcategory of **CL**, denoted **SCL**. In fact, **SCL** is a subcategory of **EACL**, since when $f = \langle f_1, f_2 \rangle : A \xrightarrow{\bullet} B$ is symmetric, then necessarily $f_1(x, x) = f_2(x, x)$ for all $x \in A$. Moreover, it is again a cartesian subcategory with the same products.

Since the first component of a symmetric morphism uniquely determines the second component, **SCL** can be represented as the category whose objects are complete lattices having as morphisms $A \xrightarrow{\bullet} B$ (where A and B are complete lattices) those functions $f : A \times A \rightarrow B$ which are monotonic in the first and anti-monotonic in the second argument. Composition, denoted \bullet , is then defined as follows. Given $f : A \xrightarrow{\bullet} B$ and $g : B \xrightarrow{\bullet} C$, $g \bullet f : A \xrightarrow{\bullet} C$ is the function

$$g \circ \langle f, f^{\text{op}} \rangle : A \times A \rightarrow C,$$

so that $h(x, x') = g(f(x, x'), f(x', x))$. The identity morphism $A \xrightarrow{\bullet} A$ is the projection $\pi_1^{A \times A}$.

5 Fixed points

In this section, we recall from [8] the construction of stable and well-founded fixed points. More precisely, only symmetric functions were considered in [8], but it was remarked that the construction also works for non-symmetric functions.

Suppose that $f = \langle f_1, f_2 \rangle : A \xrightarrow{\bullet} A$ in **CL**, so that f is a \leq_p -monotonic function $A \times A \rightarrow A \times A$. Then $f_1 : A \times A \rightarrow A$ is monotonic in its first argument and anti-monotonic in its second argument, and $f_2 : A \times A \rightarrow A$ is monotonic in its second argument and anti-monotonic in its first argument. Define the functions $s_1, s_2 : A \rightarrow A$ by

$$\begin{aligned} s_1(x') &= \mu x. f_1(x, x') \\ s_2(x) &= \mu x'. f_2(x, x') \end{aligned}$$

and let $S(f) : A \times A \rightarrow A \times A$ be the function $S(f)(x, x') = (s_1(x'), s_2(x))$. Since s_1 and s_2 are anti-monotonic, $S(f)$ is a morphism $A \xrightarrow{\bullet} A$ in **CL**. We call $S(f)$ the *stable function* for f . It is known that every fixed point of $S(f)$ is a fixed point of f , called a *stable fixed point* of f . We let f^Δ denote the set of all stable fixed points of f . Since $S(f)$ is \leq_p -monotonic, there is a \leq_p -least stable fixed point f^\ddagger , called the *well-founded fixed point* of f .

The above construction can slightly be extended. Suppose that $f = \langle f_1, f_2 \rangle : A \times B \xrightarrow{\bullet} A$ in **CL**, so that f is a function $A \times B \times A \times B \rightarrow A \times A$. Then $f_1 : A \times B \times A \times B \rightarrow A$ is monotonic in its first and second arguments and anti-monotonic in the third and fourth arguments, while $f_2 : A \times B \times A \times B \rightarrow A$ is monotonic in the third and fourth arguments and anti-monotonic in the first and second arguments. Now let $s_1, s_2 : A \times B \times B \rightarrow A$ be defined by

$$\begin{aligned} s_1(x', y, y') &= \mu x. f_1(x, y, x', y') \\ s_2(x, y, y') &= \mu x'. f_2(x, y, x', y'). \end{aligned}$$

We have that s_1 is monotonic in its second argument and anti-monotonic in the first and third arguments, and s_2 is monotonic in the third argument and anti-monotonic in the first and second arguments. Define $S(f) : A \times A \times B \times B \rightarrow A \times A$ by

$$S(f)(x, x', y, y') = (s_1(x', y, y'), s_2(x, y, y')).$$

Then $S(f)$, as a function $(A \times A) \times (B \times B) \rightarrow A \times A$, is \leq_p -monotonic in both of its arguments. We call $S(f)$ the *stable function* for f . (Note that $S(f)$ can be considered as a morphism $L \times L' \rightarrow L$ of the category **CL**, where L and L' are the complete bilattices $A \times A$ and $B \times B$ considered as complete lattices ordered by the relation \leq_p .) For each $y, y' \in B$, let $f^\Delta(y, y')$ denote the set of solutions of the fixed point equation $(x, x') = S(f)(x, x', y, y')$. Hence, f^Δ is a function from $B \times B$ to the power set of $A \times A$, that we call the *stable fixed point function*. In particular, for each $y, y' \in B$ there is a \leq_p -least element of $f^\Delta(y, y')$. We denote it by $f^\ddagger(y, y')$. Since $S(f)$ is \leq_p -monotonic, so is $f^\ddagger : B \times B \rightarrow A \times A$. Hence $f^\ddagger : B \xrightarrow{\bullet} A$ in **CL**.

We have thus defined a dagger operation ‡ on **CL**, called the (parametric) *well-founded fixed point operation*. In the next two sections, we investigate the equational properties of this operation.

Remark 5.1 The parametric well-founded fixed point operation ‡ is just the pointwise extension of the operation defined on morphisms $A \xrightarrow{\bullet} A$. Indeed, when $f : A \times B \xrightarrow{\bullet} A$ and $(y, y') \in B \times B$, then let $g : A \xrightarrow{\bullet} A$ be given by $g(x, x') = f(x, y, x', y')$. Then $f^\ddagger(y, y') = g^\ddagger$ and $f^\Delta(y, y') = g^\Delta$.

Remark 5.2 Suppose that $f : \mathbf{2} \xrightarrow{\bullet} \mathbf{2}$ is given by $f(x, x') = (\neg x', \neg x)$. Then f is symmetric but f^\ddagger is not, since $f^\ddagger = (0, 1)$. Hence **SCL** is not closed w.r.t. the parametric well-founded

fixed point operation. Let $g : \mathbf{2} \times \mathbf{2} \xrightarrow{\bullet} \mathbf{2}$ be given by $g(x, y, x', y') = (\neg x', \neg x)$. Then g is a morphism in **ACL**. However, $g^\ddagger(y, y') = (0, 1)$ for all $y, y' \in \mathbf{2}$, so that g^\ddagger is not a morphism in **ACL**. Hence, **ACL** is also not closed under the parametric well-founded fixed point operation.

Remark 5.3 We provide an example showing that when $f : A \times B \xrightarrow{\bullet} A$ in **CL** is consistent, f^\ddagger may not be consistent. Indeed, let $A = \mathbf{2}$ and $B = T$ (terminal object), and let $f : A \xrightarrow{\bullet} A$ be given by $f(x, x') = (1, \neg x \vee x')$. Then f is consistent, since $f(0, 0) = f(0, 1) = f(1, 1) = (1, 1)$, but $f^\ddagger = (1, 0)$, so that f^\ddagger is not consistent. Since f is in fact in **EACL**, this example also shows that neither **ACL** nor **EACL** is closed with respect to the well founded fixed point operation.

Note that the above f is not symmetric. In fact, if $f : A \xrightarrow{\bullet} A$ is symmetric, then $f^\ddagger : T \xrightarrow{\bullet} A$ is consistent. This follows from Remark 5.1 and Theorem 23 in [8].

6 Some valid identities

In this section we establish the parameter, fixed point, permutation and group identities and the special case (6) of the pairing identity for the parametrized well-founded fixed point operation over **CL**. In fact, we prove that the weak functorial implication holds.

Proposition 6.1 *The parameter identity holds:*

$$(f \circ (\mathbf{id}_A \times g))^\ddagger = f^\ddagger \circ g,$$

for all $f : A \times B \xrightarrow{\bullet} A$ and $g : C \xrightarrow{\bullet} B$.

Proof. Let $h = f \circ (\mathbf{id}_A \times g) : A \times C \xrightarrow{\bullet} A$. Then $S(h) : A \times A \times C \times C \rightarrow A \times A$ is given by

$$\begin{aligned} S(h)(x, x', z, z') &= (\mu x. f_1(x, g_1(z, z'), x', g_2(z, z')), \mu x'. f_1(x, g_1(z, z'), x', g_2(z, z'))) \\ &= S(f)((\mathbf{id}_{A \times A} \times g)(x, x', z, z')), \end{aligned}$$

where $f = \langle f_1, f_2 \rangle$ and $g = \langle g_1, g_2 \rangle$. Thus, $S(h) = S(f) \circ (\mathbf{id}_{A \times A} \times g)$ in **SET** (or **CL**) and therefore $h^\Delta = f^\Delta \circ (\mathbf{id}_{A \times A} \times g)$. Moreover, $h^\ddagger = f^\ddagger \circ g$, since the parameter identity holds for the least fixed point operation over **CL**. \square

Proposition 6.2 *The fixed point identity holds:*

$$f \circ \langle f^\ddagger, \mathbf{id}_B \rangle = f^\ddagger,$$

for all $f : A \times B \xrightarrow{\bullet} A$.

Proof. By Remark 5.1, it is sufficient to prove our claim only in the case when $f : A \xrightarrow{\bullet} A$, i.e., f is a \leq_p -monotonic function $A \times A \rightarrow A \times A$. But it is known that if $f : A \xrightarrow{\bullet} A$, then each stable fixed point of f is a (\leq_t -minimal) fixed point, so $f \circ f^\ddagger = f^\ddagger$. (We also have $f \circ f^\Delta = f^\Delta$.) \square

Proposition 6.3 *The permutation identity holds:*

$$(\rho \circ f \circ (\rho^{-1} \times \text{id}_B))^{\dagger} = \rho \circ f^{\dagger},$$

for all $f : A_1 \times \cdots \times A_n \times B \xrightarrow{\bullet} A_1 \times \cdots \times A_n$ and permutation $\rho : [n] \rightarrow [n]$.

Proof. We prove this only when B is the terminal object, so that f can be viewed as a morphism $f = \langle f_1, f_2 \rangle : A_1 \times \cdots \times A_n \xrightarrow{\bullet} A_1 \times \cdots \times A_n$, where f_1, f_2 are appropriate functions

$$A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n \rightarrow A_1 \times \cdots \times A_n.$$

Let $g = \rho \circ f \circ \rho^{-1}$ in **CL**, so that $g = \langle g_1, g_2 \rangle$ where g_1, g_2 are functions

$$A_{\rho(1)} \times \cdots \times A_{\rho(n)} \times A_{\rho(1)} \times \cdots \times A_{\rho(n)} \rightarrow A_{\rho(1)} \times \cdots \times A_{\rho(n)}.$$

First we show that

$$S(g) = \rho \circ S(f) \circ \rho^{-1} \tag{9}$$

in **CL**, i.e.,

$$S(g) = (\rho \times \rho) \circ S(f) \circ (\rho^{-1} \times \rho^{-1})$$

in SET (or CL). Below we will denote by x, x' n -tuples in $A_1 \times \cdots \times A_n$. Similarly, let y, y' denote n -tuples in $A_{\rho(1)} \times \cdots \times A_{\rho(n)}$. Note that if $x = (x_1, \dots, x_n) \in A_1 \times \cdots \times A_n$, then $\rho(x) = (x_{\rho(1)}, \dots, x_{\rho(n)})$ in $A_{\rho(1)} \times \cdots \times A_{\rho(n)}$. And if $y = (y_1, \dots, y_n) \in A_{\rho(1)} \times \cdots \times A_{\rho(n)}$, then $\rho^{-1}(y) = (y_{\rho^{-1}(1)}, \dots, y_{\rho^{-1}(n)})$ in $A_1 \times \cdots \times A_n$. Let

$$\begin{aligned} s_1(x') &= \mu x. f_1(x, x') \\ s_2(x) &= \mu x'. f_2(x, x'). \end{aligned}$$

Then $S(f)(x, x') = (s_1(x'), s_2(x))$. Similarly, let

$$\begin{aligned} t_1(y') &= \mu y. \rho(f_1(\rho^{-1}(y), \rho^{-1}(y'))) \\ t_2(y) &= \mu y'. \rho(f_2(\rho^{-1}(y), \rho^{-1}(y'))). \end{aligned}$$

Then $S(g)(y, y') = (t_1(y'), t_2(y))$. Since the permutation and parameter identities hold for the least fixed point operation over CL, we obtain that

$$\begin{aligned} t_1(y') &= \rho(s_1(\rho^{-1}(y'))) \\ t_2(y) &= \rho(s_2(\rho^{-1}(y))), \end{aligned}$$

proving (9). Now from (9), since the permutation identity holds for the least fixed point operation over CL, it follows that $g^{\dagger} = \rho \circ f^{\dagger}$ in **CL**. Moreover, it follows that the stable fixed points of g are of the form $(\rho(x), \rho(x'))$, where (x, x') is a stable fixed point of f . (A suggestive notation: $g^{\Delta} = \rho \circ f^{\Delta}$.) \square

We now establish a special case of the pairing identity. It will be shown later that the general form of the identity does not hold.

Proposition 6.4 *The identity (6) holds:*

$$\langle f, g \circ (\pi_2^{A \times B} \times \text{id}_C) \rangle^\dagger = \langle f^\dagger \circ \langle g^\dagger, \text{id}_C \rangle, g^\dagger \rangle,$$

where $f : A \times B \times C \xrightarrow{\bullet} A$ and $g : B \times C \xrightarrow{\bullet} B$.

Proof. It suffices to consider the case when there is no parameter. So let $f = \langle f_1, f_2 \rangle : A \times B \xrightarrow{\bullet} A$ and $g = \langle g_1, g_2 \rangle : B \xrightarrow{\bullet} B$, so that $f_1, f_2 : A \times B \times A \times B \rightarrow A$ and $g_1, g_2 : B \times B \rightarrow B$. Let $h = \langle f, g \circ \pi_2^{A \times B} \rangle : A \times B \xrightarrow{\bullet} A \times B$ in **CL**. Then h^\dagger can be constructed as follows. First consider

$$\begin{aligned} & \mu(x, y). (f_1(x, y, x', y'), g_1(y, y')) \quad \text{and} \\ & \mu(x', y'). (f_2(x, y, x', y'), g_2(y, y')). \end{aligned}$$

Since (6) and the parameter identity hold for the least fixed point operation over CL, we know that these functions can respectively be written as

$$\begin{aligned} & (\mu x. f_1(x, \mu y. g_1(y, y'), x', y'), \mu y. g_1(y, y')) \quad \text{and} \\ & (\mu x'. f_2(x, y, x', \mu y'. g_2(y, y')), \mu y'. g_2(y, y')). \end{aligned}$$

Now h^\dagger can be obtained by solving the system of equations

$$\begin{aligned} (x, x') &= (\mu x. f_1(x, \mu y. g_1(y, y'), x', y'), \mu x'. f_2(x, y, x', \mu y'. g_2(y, y'))) = S(f)((x, x'), S(g)(y, y')) \\ (y, y') &= (\mu y. g_1(y, y'), \mu y'. g_2(y, y')) = S(g)(y, y') \end{aligned}$$

for its least solution w.r.t. \leq_p . Moreover, it follows that h^Δ consists of all $((x, y), (x', y'))$ such that (y, y') is a stable fixed point of g and (x, x') is in $f^\Delta(y, y')$. In particular, since the least fixed point operation over CL satisfies (6), it holds that $h^\dagger = \langle f^\dagger \circ g^\dagger, g^\dagger \rangle$ as claimed. \square

Remark 6.5 The identity (6) has already been established in Theorem 3.11 of [27], see also the Splitting Set Theorem of [25].

Proposition 6.6 *The weak functorial dagger implication holds: for all $f : A^n \times B \xrightarrow{\bullet} A^n$ and $g : A \times B \xrightarrow{\bullet} A$ in **CL**: if $f \circ (\Delta_n \times \text{id}_B) = \Delta_n \circ g$, then $f^\dagger = \Delta_n \circ g^\dagger$.*

Proof. We spell out the proof only in the case when B is a terminal object. So let $f : A^n \xrightarrow{\bullet} A^n$ and $g : A \xrightarrow{\bullet} A$ in **CL**, say $f = \langle f_1, f_2 \rangle$ and $g = \langle g_1, g_2 \rangle$, where $f_i : A^n \times A^n \rightarrow A^n$ and $g_i : A \times A \rightarrow A$ are appropriate functions for $i = 1, 2$.

The assumption $f \circ \Delta_n = \Delta_n \circ g$ can be rephrased as

$$f_i \circ (\Delta_n \times \Delta_n) = \Delta_n \circ g_i, \quad i = 1, 2,$$

i.e.,

$$\begin{aligned} f_1(x, \dots, x, x', \dots, x') &= (g_1(x, x'), \dots, g_1(x, x')) \\ f_2(x, \dots, x, x', \dots, x') &= (g_2(x, x'), \dots, g_2(x, x')) \end{aligned}$$

for all $x, x' \in A$. Since the weak functorial dagger implication and the parameter identity hold for the least fixed point operation over CL, it follows that

$$\begin{aligned} h_1(x', \dots, x') &= (k_1(x'), \dots, k_1(x')) \\ h_2(x, \dots, x) &= (k_2(x), \dots, k_2(x)) \end{aligned}$$

where $h_1(x'_1, \dots, x'_n)$ and $h_2(x_1, \dots, x_n)$ are respectively the least solutions of

$$\begin{aligned} (x_1, \dots, x_n) &= f_1(x_1, \dots, x_n, x'_1, \dots, x'_n) \quad \text{and} \\ (x'_1, \dots, x'_n) &= f_2(x_1, \dots, x_n, x'_1, \dots, x'_n) \end{aligned}$$

and $k_1(x')$ and $k_2(x)$ denote the least solutions of

$$\begin{aligned} x &= g_1(x, x') \quad \text{and} \\ x' &= g_2(x, x'), \end{aligned}$$

so that $S(f)(x_1, \dots, x_n, x'_1, \dots, x'_n) = (h_1(x'_1, \dots, x'_n), h_2(x_1, \dots, x_n))$, moreover, $S(g)(x, x') = (k_1(x'), k_2(x))$. Consider now the equations

$$(x_1, \dots, x_n, x'_1, \dots, x'_n) = (h_1(x'_1, \dots, x'_n), h_2(x_1, \dots, x_n))$$

and

$$(x, x') = (k_1(x'), k_2(x)).$$

Since the weak functorial dagger implication and the parameter identity hold for the least fixed point operation over CL, the \leq_p -least solution of the first equation can be obtained as the $2n$ -tuple whose first n components are equal to the first component of the \leq_p -least solution of the second equation, and whose second n components are equal to the second component of the \leq_p -least solution of the second equation. This means that $f^\dagger = (\Delta_n \times \Delta_n) \circ g^\dagger$ in SET, i.e., $f^\dagger = \Delta_n \circ g^\dagger$ in CL. (It also holds that if (x, x') is a stable fixed point of g , then $(x, \dots, x, x', \dots, x')$ is a stable fixed point of f .) \square

Corollary 6.7 *The identities associated with finite groups hold for the parametrized well-founded fixed point operator over CL.*

In fact, each identity associated with a finite automaton holds.

7 Some identities that fail

Proposition 7.1 *The composition identity fails even in the following simple case:*

$$f \circ (f \circ f)^\dagger = (f \circ f)^\dagger,$$

where $f : A \xrightarrow{\bullet} A$.

Proof. Let $f : \mathbf{2} \xrightarrow{\bullet} \mathbf{2}$ be given by $f(x, x') = (\neg x', \neg x)$ (see also Remark 5.2). Then $f \circ f$ is the identity function on $\mathbf{2} \times \mathbf{2}$, hence $(f \circ f)^\dagger = (0, 0)$. On the other hand, $f \circ (f \circ f)^\dagger = (1, 1)$. \square

Proposition 7.2 *The squaring identity $(f \circ f)^\ddagger = f^\ddagger$ fails, where $f : A \xrightarrow{\bullet} A$.*

Proof. Let f be as in the previous proof. Then $(f \circ f)^\ddagger = (0, 0)$ as shown above. But $f^\ddagger = (0, 1)$. \square

Since the fixed point, parameter and permutation identities hold but the composition identity fails, the pairing identity also must fail, see [3]. We can give a direct proof.

Proposition 7.3 *The pairing identity*

$$\langle f, g \rangle^\ddagger = \langle f^\ddagger \circ \langle h^\ddagger, \mathbf{id}_C \rangle, h^\ddagger \rangle,$$

where $h = g \circ \langle f^\ddagger, \mathbf{id}_{B \times C} \rangle$ fails, where $f : A \times B \times C \xrightarrow{\bullet} A$ and $g : A \times B \times C \xrightarrow{\bullet} B$.

Proof. Let $f, g : \mathbf{2} \times \mathbf{2} \xrightarrow{\bullet} \mathbf{2}$ in **CL**, so that f and g are appropriate functions $\mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{2} \rightarrow \mathbf{2} \times \mathbf{2}$,

$$\begin{aligned} f(x, y, x', y') &= (\neg y', \neg y) \\ g(x, y, x', y') &= (\neg x', \neg x). \end{aligned}$$

Then

$$\langle f, g \rangle(x, y, x', y') = (\neg y', \neg x', \neg y, \neg x)$$

and thus $\langle f, g \rangle^\ddagger = (0, 0, 1, 1)$. On the other hand, $f^\ddagger(y, y') = (\neg y', \neg y)$, hence $h = g \circ \langle f^\ddagger, \mathbf{id}_2 \rangle$ is the identity function on $\mathbf{2} \times \mathbf{2}$ and $h^\ddagger = (0, 0)$ and $f^\ddagger \circ h^\ddagger = (1, 1)$. It follows that $\langle f^\ddagger \circ h^\ddagger, h^\ddagger \rangle = (1, 0, 1, 0)$. \square

Each of the above examples involved symmetric morphisms. We now refute the double dagger identity, but we use a non-symmetric morphism.

Proposition 7.4 *The double dagger identity fails in **CL**.*

Proof. Let $g : \mathbf{2} \times \mathbf{2} \xrightarrow{\bullet} \mathbf{2}$ be given by $g(x, y, x', y') = (\neg y', \neg x)$, and let $h = g \circ \langle \mathbf{id}_2, \mathbf{id}_2 \rangle : \mathbf{2} \xrightarrow{\bullet} \mathbf{2}$, so that $h(x, x') = (\neg x', \neg x)$. We already know that $h^\ddagger = (0, 1)$. But $g^\ddagger(y, y') = (\neg y', y)$ and $g^{\ddagger\ddagger} = (1, 0)$. \square

8 Conclusion

We extended the well-founded fixed point operation of [8, 27] to a parametric operation and studied its equational properties. We found that several of the identities of iteration theories hold for the parametric well-founded fixed point operation, but some others fail. Two interesting questions for further investigation arise. The first one concerns the *algorithmic description* of the valid identities of the well-founded fixed point operation. Does there exist an algorithm to decide whether an identity (in the language of cartesian categories equipped with a dagger operation) holds for the well-founded fixed point operation? The second one concerns the *axiomatic description* of the valid identities of the well-founded fixed point operation. These questions are relevant in connection with modular logic programming, cf. [20, 24, 25].

An alternative semantics of logic programs with negation based on an infinite domain of truth values was proposed in [26]. The infinite valued approach has been further developed in the abstract setting of ‘stratified complete lattices’ in [5, 18, 19, 14, 16]. In particular, it has been proved in [14] that the stratified least fixed point operation arising in this approach does satisfy all identities of iteration theories.

Acknowledgments The authors would like to thank Panos Rondogiannis for pointing out some of the references. The second author would like to thank the hospitality of the Institute of Informatics Gaspard Monge of Université Paris Est.

References

- [1] H. Bekić: Definable operations in general algebras, and the theory of automata and flowcharts. Technical report, IBM Vienna, 1969. Reprinted in: *Programming Languages and Their Definition - Hans Bekić (1936-1982)*, LNCS 177, pp. 30–55, Springer, 1984.
- [2] S.L. Bloom, C.C. Elgot and J.B. Wright: Solutions of the iteration equation and extensions of the scalar iteration operation, *SIAM J. Comput.*, 9(1980), 25–45.
- [3] S.L. Bloom and Z. Ésik: *Iteration theories*. Springer, 1993.
- [4] S.L. Bloom and Z. Ésik: Fixed-point operations on ccc’s. Part I. *Theor. Comput. Sci.* 155(1996), 1–38.
- [5] A. Charalambidis, Z. Ésik and P. Rondogiannis: Minimum model semantics for extensional higher-order logic programming with negation, *Theor. Prac. Log. Prog.*, 14(2014), 725–737.
- [6] B.A. Davey and H.A. Priestley: *Introduction to lattices and order*, 2nd Edition, Cambridge University Press, 2002.
- [7] J.W. De Bakker and D. Scott: A theory of programs. Technical Report, IBM Vienna, 1969.
- [8] M. Denecker, V.M. Marek and M. Truszczyński: Approximations, stable operators, well-founded fixpoints and applications in nonmonotonic reasoning, in: *Logic-Based Artificial Intelligence*, Springer International Series in Engineering and Comput. Sci., Vol. 597, Chapter 6, pp. 127–144, 2000.
- [9] M. Denecker, V.M. Marek and M. Truszczyński: Ultimate approximation and its applications in nonmonotonic knowledge representation systems, *Inform. Comput.*, 192(2004), 82–121.
- [10] Z. Ésik: Identities in iterative and rational algebraic theories. *Comput. Linguist. Comput. Lang.*, 14(1980), 183–207.
- [11] Z. Ésik: Group axioms for iteration. *Inform. Comput.*, 148(1999), 131–180.
- [12] Z. Ésik: Axiomatizing iteration categories, *Acta Cybern.*, 14(1999), 65–82.
- [13] Z. Ésik: The power of the group-identities for iteration, *Int. J. Algebra Comp.*, 10(2000), 349–374.

- [14] Z. Ésik: Equational properties of stratified least fixed points (Extended abstract), in: *WoLLIC 2015*, LNCS 9160, pp. 174–188, 2015.
- [15] Z. Ésik: Equational axioms associated with finite automata for fixed point operations in cartesian categories, *Math. Struct. Comput. Sci.*, to appear.
- [16] Z. Ésik: A representation theorem for stratified complete lattices, CoRR abs/1503.05124, 2015.
- [17] Z. Ésik: Equational properties of fixed point operations in cartesian categories: An overview. In: *MFCS (1)*, Springer, LNCS 9234, pp. 18–37, 2015.
- [18] Z. Ésik and P. Rondogiannis: Theorems on pre-fixed points of non-monotonic functions with applications in logic programming and formal grammars, *WoLLIC 2014*, LNCS 8652, pp. 166–180, 2014.
- [19] Z. Ésik and P. Rondogiannis: A fixed point theorem for non-monotonic functions, *Theor. Comput. Sci.*, 574(2015), 18–38.
- [20] P. Ferraris, J. Lee, V. Lifschitz and R. Palla: Symmetric splitting in the general theory of stable models in: proc. *IJCAI 2009*, pp. 797–803, IJCAI Organization, 2009.
- [21] M. Fitting: Fixed point semantics for logic programming, a survey, *Theoret. Comput. Sci.*, 278(2002), 25–51.
- [22] M. Fitting: Bilattices are nice things, in: *Self-Reference*, Center for the Study of Language and Information, pp. 53–77, 2006.
- [23] M. Ginsberg: Multivalued logics: a uniform approach to reasoning in AI. *Comput. Intelligence*, 4(1988), 256–316.
- [24] T. Janhunen, E. Oikarinen, H. Tompits and S. Woltran: Modularity aspects of disjunctive stable models, *J. Artificial Intell. Research*, 35(2009), 813–857.
- [25] V. Lifschitz and H. Turner: Splitting a logic program, in: proc. *Logic Programming 1994*, pp. 23–37, MIT Press, 1994.
- [26] P. Rondogiannis and W.W. Wadge: Minimum model semantics for logic programs with negation-as-failure, *ACM Trans. Comput. Log.*, 6(2005), 441–467.
- [27] J. Vennekens, D. Gilis and M. Denecker: Splitting an operator: Algebraic modularity results for logics with fixpoint semantics, *ACM Transactions on Computational Logic*, 5(2009), 1–32.